

# From $sl_q(2)$ to a Parabosonic Hopf Algebra

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**Abstract.** A Hopf algebra with four generators among which an involution (reflection) operator, is introduced. The defining relations involve commutators and anticommutators. The discrete series representations are developed. Designated by  $sl_{-1}(2)$ , this algebra encompasses the Lie superalgebra  $osp(1|2)$ . It is obtained as a  $q = -1$  limit of the  $sl_q(2)$  algebra and seen to be equivalent to the parabosonic oscillator algebra in irreducible representations. It possesses a noncocommutative coproduct. The Clebsch–Gordan coefficients (CGC) of  $sl_{-1}(2)$  are obtained and expressed in terms of the dual  $-1$  Hahn polynomials. A generating function for the CGC is derived using a Bargmann realization.

*Key words:* parabosonic algebra; dual Hahn polynomials; Clebsch–Gordan coefficients

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## 1 Introduction

On the one hand, algebraic structures are natural descriptors of symmetries. On the other, the exact solutions of the dynamical equations of physical systems, when they exist, are typically presented in terms of special functions and orthogonal polynomials. Not surprisingly hence, the relations between solvable models, special functions, symmetries and their algebraic translations is of considerable interest.

The presence of reflection operators has been seen to arise in many contexts, physical and mathematical, related in particular, to the first two of the above areas. To give some examples, recall that in integrable many-body problems of the Calogero type, operators with reflections play a key role in expressing the constants of motion that are in involution [10, 2]. There is currently much activity also in the study of Dunkl harmonic analysis [17].

Recently, we have examined univariate polynomials that are eigenfunctions of operators of Dunkl type, that is of operators that are first order in the derivative and involve reflections. We have thus discovered certain families of “classical” orthogonal polynomials that had hitherto escaped notice [23, 24].

It has been found that these polynomials can be identified as a  $q \rightarrow -1$  limits of some  $q$ -orthogonal polynomials, the simplest among them being the little  $-1$  Jacobi polynomials introduced in [22].

In [18] and [19] this approach was generalized to Dunkl shift operators. This provided a theoretical framework for the Bannai–Ito and the dual  $-1$  Hahn polynomials.

With this perspective, it is thus natural to examine algebraic structures involving reflection operators and it is the purpose of this paper to contribute to such a study. For related investigations see, e.g. [1, 8, 6, 7].

## 2 Definition of the $sl_{-1}(2)$ algebra and its relation with the $osp(1|2)$ Lie superalgebra

We define  $sl_{-1}(2)$  as the algebra which is generated by the four elements  $J_0$ ,  $J_{\pm}$  and  $R$  subject to the relations

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [R, J_0] = 0, \quad \{J_+, J_-\} = 2J_0, \quad \{R, J_{\pm}\} = 0, \quad (2.1)$$

where  $[A, B] = AB - BA$  and  $\{A, B\} = AB + BA$ . The operator  $R$  is an involution operator, i.e. it satisfies the property

$$R^2 = I.$$

The Casimir operator  $Q$ , which by definition commutes with all the generators  $(R, J_0, J_{\pm})$ , is

$$Q = J_+ J_- R - (J_0 - 1/2) R. \quad (2.2)$$

Like the ordinary  $sl(2)$  or its quantum analogue  $sl_q(2)$ , the algebra  $sl_{-1}(2)$  possesses a non-trivial discrete series representation.

Indeed, let  $e_n$ ,  $n = 0, 1, 2, \dots$  denote the basis vectors, and define the action of the operators by the formulas

$$J_0 e_n = (n + \mu + 1/2) e_n, \quad J_- e_n = \rho_n e_{n-1}, \quad J_+ e_n = \rho_{n+1} e_{n+1},$$

where  $\mu$  is a constant and  $\rho_n$  are the positive matrix elements of the representation. Moreover, demand that  $\rho_0 = 0$  in order to obtain the standard discrete series bounded from below and with  $n = 0, 1, 2, \dots$ .

The operator  $R$  commutes with  $J_0$  and hence can be diagonalized in the basis  $e_n$ . A simple analysis based on the properties of  $R$ , leads to the conclusion that

$$R e_n = \epsilon (-1)^n e_n, \quad n = 0, 1, 2, \dots, \quad (2.3)$$

where  $\epsilon = \pm 1$  is a fixed parameter in a given representation.

Expressing the commutation relations in the basis  $e_n$  gives the following equation for  $\rho_n$

$$\rho_n^2 + \rho_{n+1}^2 = 2(n + \mu + 1/2)$$

with general solution

$$\rho_n^2 = n + \mu + \kappa (-1)^n,$$

where  $\kappa$  is an arbitrary constant.

The condition  $\rho_0 = 0$  means that  $\kappa = -\mu$  and we thus have

$$\rho_n^2 = n + \mu(1 - (-1)^n).$$

The Casimir operator (2.2), as should be, is a multiple of the identity operator

$$Q e_n = -\epsilon \mu e_n$$

on the module with the basis  $\{e_n\}$ .

The matrix elements can be presented in the form

$$\rho_n^2 = n + \mu(1 - (-1)^n) = [n]_\mu,$$

in terms of the “mu-numbers”

$$[n]_\mu = n + \mu(1 - (-1)^n). \quad (2.4)$$

We define also the “mu-factorials” by

$$[n]_\mu! = [1]_\mu [2]_\mu [3]_\mu \cdots [n]_\mu.$$

If we assume that

$$\mu > -1/2$$

then  $\rho_n^2 > 0$  for  $n = 1, 2, 3, \dots$ , and we thus obtain a unitary infinite-dimensional representation of the algebra  $sl_{-1}(2)$ . The value of the Casimir operator is  $Q = -\epsilon\mu$  in this representation.

Thus, the discrete series representation is fixed by two parameters  $\epsilon = \pm 1$  and  $\mu > -1/2$ .

Let us now indicate the connection that  $sl_{-1}(2)$  has with the simplest Lie superalgebra  $osp(1|2)$ . Consider the elements  $K_\pm = J_\pm^2$ . It is easy to verify that  $J_0$ ,  $K_+$  and  $K_-$  satisfy together the commutation relations of the  $sl(2)$  algebra

$$[K_-, K_+] = 4J_0, \quad [J_0, K_\pm] = \pm 2K_\pm.$$

Hence,  $J_0$ ,  $J_\pm$ ,  $K_\pm$  form a basis for the Lie superalgebra  $osp(1|2)$  [4]. The operators  $J_0$ ,  $K_\pm$  belong to the even part of this algebra, while the operators  $J_\pm$  belong to the odd part.

The Casimir operator (2.2) of the  $sl_{-1}(2)$  algebra contains the involution operator  $R$  which commutes with the operators  $J_0$  and  $J_+J_-$ . Hence the square  $Q^2$  of the Casimir operator will commute with all the generators of the  $sl_{-1}(2)$  algebra. However its expression will contain only the operators  $J_0$ ,  $J_\pm$  and not  $R$ :

$$Q^2 = (J_0 - 1/2)^2 - J_+^2 J_-^2 - J_+ J_- = (J_0 - 1/2)^2 - K_+ K_- - J_+ J_-.$$

This operator coincides with the Casimir operator of the Lie superalgebra  $osp(1|2)$  [4]. We see that the Casimir operator  $Q$  of the algebra  $sl_{-1}(2)$  can be considered as a “square root” of the Casimir operator for the algebra  $osp(1|2)$ .

In the next section we show that the algebra  $sl_{-1}(2)$  can be obtained as a  $q \rightarrow -1$  limit of the algebra  $sl_q(2)$ . This justifies the name of the algebra.

### 3 The $sl_{-1}(2)$ algebra as a limit of the $sl_q(2)$ algebra

Consider the algebra generated by three operators  $J_0$ ,  $J_\pm$ , with commutation relations [3]

$$[J_0, J_\pm] = \pm J_\pm, \quad J_- J_+ - q J_+ J_- = 2 \frac{q^{2J_0} - 1}{q^2 - 1}, \quad (3.1)$$

where  $q$  is a real parameter.

The Casimir operator  $Q$ , commuting with  $J_0$  and  $J_\pm$  is

$$Q = J_+ J_- q^{-J_0} - \frac{2}{(q^2 - 1)(q - 1)} (q^{J_0 - 1} + q^{-J_0}).$$

In what follows we restrict ourselves to discrete series representations of the algebra (3.1). This means representations that have bases  $e_n$ ,  $n = 0, 1, \dots$  such that

$$J_0 e_n = (n + \nu) e_n, \quad J_- e_n = r_n e_{n-1}, \quad J_+ e_{n+1} = r_{n+1} e_{n+1}.$$

As usual, the condition  $r_0 = 0$  is assumed. It is easily verified that

$$r_n^2 = \frac{2(1 - q^n)(1 - q^{n+2\nu-1})}{(q+1)(q-1)^2}.$$

The parameter  $\nu$  is related to the value of the Casimir operator

$$Q = \frac{2(q^{\nu-1} + q^{-\nu})}{(1-q)(q^2-1)}$$

in these representations. The Fock–Bargmann realization of the algebra (3.1) can be defined on the space of polynomials in the variable  $z$  by the formulas:

$$J_0 = z\partial_z + \nu, \quad J_+ = z, \quad J_- = \alpha z D_q^2 + \beta D_q, \quad (3.2)$$

where

$$\alpha = \frac{2q^{2\nu}}{1+q}, \quad \beta = \frac{2(1-q^{2\nu})}{1-q^2}$$

and  $D_q$  is the standard  $q$ -derivative operator

$$D_q f(z) = \frac{f(zq) - f(z)}{z(q-1)}.$$

In this realization the basis vectors  $e_n(z)$  are the monomials  $e_n(z) = \gamma_n z^n$ , where

$$\gamma_n = \frac{1}{\sqrt{r_1 r_2 \cdots r_n}}$$

is the normalization coefficient.

When  $0 < q < 1$ , the algebra defined by (3.1) is equivalent to the quantum  $sl_q(2)$  algebra defined by the relations

$$[A_0, A_{\pm}] = \pm A_{\pm}, \quad [A_-, A_+] = 2 \frac{q^{A_0} - q^{-A_0}}{q - q^{-1}}.$$

Indeed, under the identifications

$$J_+ = A_+ q^{(A_0-1)/2}, \quad J_- = q^{(A_0-1)/2} A_-, \quad J_0 = A_0, \quad (3.3)$$

the commutation relations (3.1) are transformed into the commutation relations (3.3).

When  $q \rightarrow 1$  the algebra  $sl_q(2)$  with the defining relations (3.1) becomes the  $sl(2)$  algebra:

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_-, J_+] = 2J_0.$$

There is also a nontrivial limit when  $q \rightarrow -1$ . It is obvious that the commutation relations (3.1) become the commutation relations (2.1) when  $q = -1$ . The limit process for the matrix coefficients  $r_n$  is more subtle however.

Assume that  $\nu = j = 1, 2, 3, \dots$  is a positive integer. Let  $q = -e^\tau$ , then the limit  $q \rightarrow -1$  is equivalent to the limit  $\tau \rightarrow 0$ .

Assume first that  $n = 0, 2, 4, \dots$  is even. Then

$$r_n^2 = \frac{2(1 - (-1)^n e^{\tau n})(1 - (-1)^{n+2\nu-1} e^{\tau(n+2\nu-1)})}{(1 - e^\tau)(1 + e^\tau)^2} = \frac{2(1 - e^{\tau n})(1 + e^{\tau(n+2\nu-1)})}{(1 - e^\tau)(1 + e^\tau)^2}.$$

Hence

$$\lim_{q \rightarrow -1} r_n^2 = \lim_{\tau \rightarrow 0} \frac{1 - e^{\tau n}}{1 - e^\tau} = n.$$

When  $n$  is odd, we have

$$r_n^2 = \frac{2(1 + e^{\tau n})(1 - e^{\tau(n+2\nu-1)})}{(1 - e^\tau)(1 + e^\tau)^2},$$

hence

$$\lim_{q \rightarrow -1} r_n^2 = \lim_{\tau \rightarrow 0} \frac{1 - e^{\tau(n+2\nu-1)}}{1 - e^\tau} = n + 2\nu - 1,$$

and

$$\lim_{q \rightarrow -1} r_n^2 = n + \mu(1 - (-1)^n) = [n]_\mu = \rho_n^2$$

where  $\nu = \mu + 1/2$ .

Thus, for integer values of the parameter  $\nu$  the limit  $q \rightarrow -1$  of the matrix elements  $r_n$  gives the expected matrix elements  $\rho_n$  of the discrete series of the  $sl_{-1}(2)$  algebra.

When  $\nu$  is not an integer, the limit of  $r_n$  is not well defined. In this case we can assume that the limiting matrix element  $\rho_n^2$  is obtained by a linear interpolation from the integer  $\nu$  case.

If  $\nu = j$  is integer, the involution operator  $R$  can also be obtained in the limit  $q \rightarrow -1$

$$R = \lim_{q \rightarrow -1} q_0^J.$$

Indeed, we have

$$q^{J_0} e_n = q^{n+j} e_n.$$

So, in the limit

$$R e_n = \epsilon(-1)^n e_n,$$

where

$$\epsilon = \lim_{q \rightarrow -1} q^j = (-1)^j = \pm 1.$$

This uniquely characterizes the involution operator with the property  $R^2 = I$ .

We thus see that the generators  $J_0, J_{\pm 1}$  and  $R$  of  $sl_{-1}(2)$  can be obtained from the algebra (3.1) when the representation parameter is a positive integer  $\nu = \mu = 1, 2, 3, \dots$ . If  $\nu$  is a real positive parameter, then the limiting process is not well defined and we postulate that in the limit  $q \rightarrow -1$  the matrix elements  $\rho_n$  correspond to the matrix elements  $r_n$  with  $\nu$  real and positive.

Note that the  $q \rightarrow -1$  limit considered here is different from the well known special case of  $sl_q(2)$  for  $q$  a root of unity [15]. In the latter case the operators  $J_{\pm}$  are nilpotent  $J_{\pm}^N = 0$ , where  $N$  is the order of the root of unity and hence all irreducible representations are restricted to be of dimension  $N$ . In our case we have infinite-dimensional representations.

## 4 Relation with the parabosonic oscillator and the Fock–Bargmann realization

Consider the commutator  $[J_-, J_+]$ . We have

$$[J_-, J_+] = \{J_-, J_+\} - 2J_+J_- = 2J_0 - 2J_+J_-.$$

Remembering the expression (2.2) for the Casimir operator, we find that

$$[J_-, J_+] = 1 - 2QR.$$

For representations with a fixed value  $\epsilon = \pm 1$ , we have  $Q = -\epsilon\mu$  and hence

$$[J_-, J_+] = 1 + 2\epsilon\mu R. \quad (4.1)$$

This relation, (4.1), defines the parabosonic oscillator algebra [20, 12, 11, 16] with operators  $J_-$ ,  $J_+$ ,  $R$  satisfying the commutation relations (4.1) and  $\{R, J_{\pm}\} = 0$  together with the condition  $R^2 = I$ .

Conversely, assume that the operators  $J_-$ ,  $J_+$ ,  $R$  form a representation of the parabosonic oscillator algebra. We can define the operator  $J_0$  as  $J_0 = \{J_+, J_-\}/2$ . Then it is easily verified that the operators  $J_0$ ,  $J_+$ ,  $J_-$ ,  $R$  satisfy the relations (2.1) defining the  $sl_{-1}(2)$  algebra.

Thus, if one restricts to irreducible representations with a fixed value of the Casimir operator  $Q = -\epsilon\mu$ , the algebra  $sl_{-1}(2)$  is equivalent to the parabosonic oscillator algebra.

For definiteness, in what follows we will use representations for which  $\epsilon = 1$ .

We can construct the Fock–Bargmann representation of the  $sl_{-1}(2)$  algebra in terms of first order differential-difference operators. Indeed, one can use the well known realization of the parabosonic operators [12, 16]

$$R = R_x, \quad J_+ = x, \quad J_- = \partial_x + \frac{\mu}{x}(1 - R_x), \quad J_0 = x\partial_x + \mu + 1/2, \quad (4.2)$$

where  $R_x$  is the reflection (parity) operator defined by  $Rf(x) = f(-x)$  for every function  $f(x)$ .

The operator  $J_-$  coincides in this realization with the standard Dunkl operator [2].

Note that when  $\nu$  is integer, the realization (4.2) can be obtained as a limit  $q \rightarrow -1$  from the realization (3.2).

The basis  $e_n(x)$  is here realized by the monomials

$$e_n(x) = \gamma_n x^n,$$

with some constants  $\gamma_n$ . If we take

$$\gamma_n = \frac{1}{\sqrt{[n]_{\mu}!}},$$

we reproduce the canonical formulas of the previous section

$$J_- e_n(x) = \rho_n e_{n-1}, \quad J_+ e_n(x) = \rho_{n+1} e_{n+1}(x).$$

Sometimes it will be convenient to take  $\gamma_n = 1$ , i.e.  $e_n(x) = x^n$ . In this case we have

$$J_0 e_n(x) = (n + \mu + 1/2) e_n(x), \quad J_- e_n(x) = [n]_{\mu} e_{n-1}(x), \quad J_+ e_n(x) = e_{n+1}(x).$$

Of course, the Casimir operator reduces (up to a constant factor) to the identity operator

$$Q e_n(x) = -\mu e_n(x).$$

Note that similar relations were investigated in [5, 13]. Our approach is different, because we start from the algebra  $sl_{-1}(2)$  with 4 generators which is observed to be a limiting case of the  $sl_q(2)$  algebra. The relation (in irreducible representations) with the parabose algebra is obtained a posteriori.

## 5 Coproduct and the Clebsch–Gordan coefficients

The most important property of the  $sl_{-1}(2)$  algebra is that it admits an “addition rule”, or a coproduct which can be inferred from the well known coproduct of the quantum algebra  $sl_q(2)$ .

Assume that we have two independent representations of the algebra (3.1) on the linear spaces  $S_1$  and  $S_2$ . Let  $S_1 \otimes S_2$  be the direct product of these spaces. We will denote by  $A \otimes B$ , the direct products of operators acting on the spaces  $S_1$  and  $S_2$ ,  $A \in \text{End}(S_1)$ ,  $B \in \text{End}(S_2)$ . It is readily verified that the elements

$$\tilde{J}_0 = J_0 \otimes I + I \otimes J_0, \quad \tilde{J}_{\pm} = J_{\pm} \otimes q^{J_0} + I \otimes J_{\pm}$$

again satisfy the commutation relations (3.1) of the  $sl_q(2)$  algebra [3]. (Here  $I$  stands for the identity operator).

Assuming that the representation parameter  $\nu$  is a positive integer, we have a well-defined  $q \rightarrow -1$  limit from  $sl_q(2)$  to  $sl_{-1}(2)$ . The operator  $q^{J_0}$  in this limit becomes  $\epsilon R$  with  $\epsilon = \pm 1$ . It is thus natural to expect that for arbitrary representation parameter  $\mu > -1/2$ , the  $sl_{-1}(2)$  algebra admits a coproduct rule.

It can be defined as follows. For two independent representations of the  $sl_{-1}(2)$  algebra with the Casimir parameters  $\mu_1, \mu_2$ , let us introduce the following operators  $\tilde{J}_0, \tilde{J}_{\pm}, \tilde{R}$  that act on the direct product of the spaces  $S_1, S_2$ :

$$\tilde{J}_0 = J_0 \otimes I + I \otimes J_0, \quad \tilde{J}_{\pm} = J_{\pm} \otimes R + I \otimes J_{\pm}, \quad \tilde{R} = R \otimes R. \quad (5.1)$$

Then the operators  $\tilde{J}_0, \tilde{J}_{\pm}, \tilde{R}$  satisfy the commutation relations (2.1), i.e. they are again generators of the algebra  $sl_{-1}(2)$ . The verification of this statement is elementary.

Note that a similar coproduct was proposed in [1, 8] for the parabosonic oscillator algebra, in the identification of its Hopf algebra structure.

In what follows we restrict ourselves to representations with  $\epsilon_1 = \epsilon_2 = 1$  and  $\mu_1 > -1/2$ ,  $\mu_2 > -1/2$ .

In the Fock–Bargmann realization,  $S_1$  and  $S_2$  are spaces of polynomials in the arguments, say,  $x$  and  $y$ . We define representations with the parameters  $\mu_1$  and  $\mu_2$  on these spaces by the formulas

$$J_0^{(x)} = x\partial_x + \mu_1 + 1/2, \quad J_+^{(x)} = x, \quad J_-^{(x)} = \partial_x + \frac{\mu_1}{x}(1 - R_x)$$

and

$$J_0^{(y)} = y\partial_y + \mu_2 + 1/2, \quad J_+^{(y)} = y, \quad J_-^{(y)} = \partial_y + \frac{\mu_2}{y}(1 - R_y).$$

The Casimir operators take the constant values  $Q_1 = -\mu_1$ ,  $Q_2 = -\mu_2$  on these representations.

Following (5.1), the operators of the coproduct are given as

$$\begin{aligned} \tilde{J}_0 &= x\partial_x + y\partial_y + \mu_1 + \mu_2 + 1, & \tilde{J}_+ &= xR_y + y, \\ \tilde{J}_- &= (\partial_x + \mu_1 x^{-1}(1 - R_x))R_y + y\partial_y + \mu_2 y^{-1}(1 - R_y). \end{aligned}$$

The corresponding Casimir operator

$$\tilde{Q} = \tilde{J}_+ \tilde{J}_- \tilde{R} - (\tilde{J}_0 - 1/2) \tilde{R}$$

commutes with the “local” Casimir operators  $Q_1$  and  $Q_2$  and with the operators  $\tilde{J}_0, \tilde{J}_{\pm}$  but not with the operators  $J_0^{(x)}, J_0^{(y)}$ .

Hence we can posit the Clebsch–Gordan problem as follows.

In view of (5.1), the operator  $\tilde{J}_0$  can take the eigenvalues  $\mu_1 + \mu_2 + N + 1$ , where  $N = 0, 1, 2, \dots$ . We denote by  $\Phi_{N,q}$ , the eigenstate with fixed eigenvalues of the total Casimir operator and of  $\tilde{J}_0$ :

$$\tilde{Q}\Phi_{N,k} = q_k\Phi_{N,k}, \quad \tilde{J}_0\Phi_{N,k} = (\mu_1 + \mu_2 + N + 1)\Phi_{N,k}.$$

This state can be decomposed as a linear combination of direct product of states:

$$\Phi_{N,k} = \sum_{s=0}^N W_{s;N,k} e_s \otimes e_{N-s}, \quad (5.2)$$

with coefficients  $W_{s;N,k}$  that can be called the Clebsch–Gordan coefficients of the  $sl_{-1}(2)$  algebra.

It is not difficult to see that the Casimir eigenvalue  $q_k$  has the expression

$$q_k = (-1)^{k+1}(\mu_1 + \mu_2 + 1/2 + k), \quad k = 0, 1, \dots, N. \quad (5.3)$$

Indeed, the eigenvalues of the operator  $J_0$  are  $n + \mu + 1/2 = n - \epsilon Q + 1/2$  (recall that  $Q = -\epsilon\mu$  in the given representation). Hence, if the eigenvalue  $\lambda > 0$  of  $J_0$  is fixed, then the possible eigenvalues of the Casimir operator in absolute value are:

$$|Q| = |\lambda - 1/2|, |\lambda - 3/2|, \dots \quad (5.4)$$

When considering the coproduct of two  $sl_{-1}(2)$  algebras, we know that the eigenvalues of  $\tilde{J}_0$  have the form  $\tilde{\lambda} = \mu_1 + \mu_2 + N + 1$ . Hence, from (5.4) we have for the set of absolute values (recall that the total number of eigenvalues should be equal to  $N + 1$ )

$$|\tilde{Q}| = \mu_1 + \mu_2 + N + 1/2, \mu_1 + \mu_2 + N - 1/2, \dots, \mu_1 + \mu_2 + 1/2.$$

The state with the maximal absolute value  $|q_N| = \mu_1 + \mu_2 + N + 1/2$  of the Casimir operator  $\tilde{Q}$ , corresponds to the state  $\tilde{e}_0$  satisfying the conditions:

$$\tilde{J}_0\tilde{e}_0 = (\mu_1 + \mu_2 + N + 1)\tilde{e}_0, \quad \tilde{J}_-\tilde{e}_0 = 0.$$

In order to determine the sign of the eigenvalue  $q_N$ , we notice that

$$\tilde{R}\Phi_{N,k} = (R_1 \otimes R_2)\Phi_{N,k} = (-1)^N \Phi_{N,k}. \quad (5.5)$$

This means on the one hand that

$$\tilde{R}\tilde{e}_0 = (-1)^N \tilde{e}_0.$$

On the other hand, by (2.3)  $\tilde{R}\tilde{e}_0 = \tilde{\epsilon}\tilde{e}_0$  and hence  $\tilde{\epsilon} = (-1)^N$ , where  $\tilde{\epsilon}$  stands for the eigenvalue of the parity operator  $\tilde{R}$  on the state  $\tilde{e}_0$ . We thus have

$$q_N = (-1)^{N+1}(\mu_1 + \mu_2 + N + 1/2).$$

Taking into account the parity of the coproduct states we arrive at formula (5.3).

In order to find the coefficients  $W_{s;N,k}$  we shall derive a 3-term recurrence relation for them.

Taking into account relation (5.5), we see that the eigenvalue equation  $\tilde{Q}\Phi_{N,k} = q_k\Phi_{N,k}$  can be presented in the form

$$Q_0\Phi_{N,k} = (-1)^N q_k\Phi_{N,k},$$



where

$$Q_0 = \tilde{J}_+ \otimes \tilde{J}_- - \tilde{J}_0 + 1/2.$$

From the expression for the Casimir operator it is seen that  $Q_0$  is tri-diagonal in the basis  $e_s \otimes e_{N-s}$ . Hence, the Clebsch–Gordan coefficients  $W_{s;N,k}$  satisfy the 3-term recurrence relation

$$A_{s+1}W_{s+1;N,k} + A_sW_{s-1;N,k} + B_sW_{s;N,k} = (-1)^N q_k W_{s;N,k},$$

where the recurrence coefficients  $A_s, B_s$  are easily expressed in terms of the known representation matrix elements for  $sl_{-1}(2)$ :

$$A_s = (-1)^s \sqrt{[s]_{\mu_1} [N - s + 1]_{\mu_2}}$$

and

$$B_s = (-1)^N ([s]_{\mu_1} + [N - s]_{\mu_2} - N - \mu_1 - \mu_2 - 1/2),$$

where we adopt the notation (2.4).

Note that the expression for the coefficient  $B_s$  can be simplified to:

$$B_s = \begin{cases} -\frac{1}{2} - (-1)^s(\mu_1 + \mu_2) & \text{if } N \text{ even,} \\ \frac{1}{2} + (-1)^s(\mu_1 - \mu_2) & \text{if } N \text{ odd.} \end{cases}$$

Thus the CGC are expressed in terms of some orthogonal polynomials  $P_s(x)$

$$W_{s;N,k} = W_{0;N,k} P_s(q_k; N). \quad (5.6)$$

These orthogonal polynomials satisfy the 3-term recurrence relation

$$A_{s+1}P_{s+1}(x) + A_sP_{s-1}(x) + B_sP_s(x) = xP_s(x)$$

with initial conditions  $P_{-1} = 0, P_0 = 1$ . From the above expressions for  $A_s, B_s$  we can conclude that the polynomials  $P_s(x)$  coincide with the generic dual  $-1$  Hahn polynomials [19].

Indeed, it is convenient to present the polynomials  $P_n(x)$  in monic form

$$P_n(x) = \frac{\hat{P}_n(x)}{A_1 A_2 \cdots A_n}.$$

Then the polynomials  $\hat{P}_n(x) = x^n + O(x^{n-1})$  satisfy on the one hand

$$\hat{P}_{n+1}(x) + u_n \hat{P}_n(x) + B_n \hat{P}_n(x) = x \hat{P}_n(x),$$

where

$$u_n = A_n^2 = [n]_{\mu_1} [N - n + 1]_{\mu_2}.$$

Note that  $u_n > 0, n = 1, 2, \dots, N$  and  $u_{N+1} = 0$ .

On the other hand, the dual  $-1$  Hahn polynomials [19]  $R_n^{(-1)}(x; \alpha, \beta; N)$  depend on 3 parameters  $\alpha, \beta$  and  $N = 1, 2, \dots$  and obey the recurrence relation

$$R_{n+1}^{(-1)}(x) + u_n^{(-1)} R_{n-1}^{(-1)}(x) + b_n^{(-1)} R_n^{(-1)}(x) = x R_n^{(-1)}(x),$$

where the recurrence coefficients are [19]

$$u_n = 4[n]_{\xi} [N + 1 - n]_{\eta}, \quad b_n = 2([n]_{\xi} + [N - n]_{\eta}) + \zeta.$$

The parameters  $\xi, \eta, \zeta$  are related to the parameters  $\alpha, \beta, N$ . When  $N$  is even

$$\xi = \frac{\beta - N - 1}{2}, \quad \eta = \frac{\alpha - N - 1}{2}, \quad \zeta = 1 - \alpha - \beta. \quad (5.7)$$

When  $N$  is odd

$$\xi = \alpha/2, \quad \eta = \beta/2, \quad \zeta = -2N - 1 - \alpha - \beta. \quad (5.8)$$

Comparing the recurrence coefficients of the polynomials  $\hat{P}_n(x)$  with the corresponding coefficients of the dual  $-1$  Hahn polynomials we conclude that

$$\hat{P}_n(x) = 2^{-n} R_n^{(-1)}(2(x - x_0); \alpha, \beta, N),$$

where the parameters  $\alpha, \beta$  are found from formulas (5.7) and (5.8) with  $\xi = \mu_1, \eta = \mu_2$ . The shift parameter  $x_0$  can also be expressed in terms of  $\mu_1, \mu_2$  in an obvious way.

We thus expressed the Clebsch–Gordan coefficients of the  $sl_{-1}(2)$  algebra in terms of the dual  $-1$  Hahn polynomials  $R_n^{(-1)}(x; \alpha, \beta; N)$ .

The remaining problem is to find an explicit expression for the coefficient  $W_{0;N,k}$  in (5.6). This can be done using the following observation. The vectors  $\psi_s = e_s \otimes e_{N-s}$  form an orthonormal basis in the  $N + 1$ -dimensional linear space. There is thus a scalar product such that

$$(\psi_s, \psi_t) = \delta_{st}.$$

The vectors  $\Phi_{N,k}$  form another orthonormal basis on the same space and so:

$$(\Phi_{N;k}, \Phi_{N;l}) = \delta_{kl}.$$

Hence, the matrix  $W_{s;N,k}$  is orthogonal, i.e. it obeys

$$\sum_{k=0}^N W_{n;N,k} W_{m;N,k} = \delta_{nm}.$$

Taking into account formula (5.6) we thus have on the one hand

$$\sum_{k=0}^N W_{0;N,k}^2 P_n(q_k) P_m(q_k) = \delta_{nm}.$$

On the other hand, the orthonormal dual  $-1$  Hahn polynomials  $P_n(x)$  satisfy the orthogonality property [19]

$$\sum_{k=0}^N w_k P_n(q_k) P_m(q_k) = \delta_{nm},$$

where  $w_k$  are positive discrete weights (concentrated masses) localized at the spectral points  $q_k$ . (The positivity property  $w_k > 0$  follows from the positivity of the recurrence coefficients  $u_n > 0$ ,  $n = 1, 2, \dots, N$  [19].)

We thus have

$$W_{0;N,k} = \sqrt{w_k}.$$

Explicit expressions for the weights were found in [19]. This solves the problem of finding the Clebsch–Gordan coefficients  $W_{s;N,k}$  up to sign factors  $\pm 1$ .

The result is not surprising. We have seen that the  $sl_{-1}(2)$  algebra is a  $q \rightarrow -1$  limit of the  $sl_q(2)$  algebra and for the latter algebra, the CGC are expressed in terms of the dual  $q$ -Hahn polynomials [9].

Also, when  $\mu_1 = \mu_2 = 0$  the dual  $-1$  Hahn polynomials coincide with the ordinary Krawtchouk polynomials. This result is also expected: the case  $\mu_1 = \mu_2 = 0$  corresponds to the case when both  $sl_{-1}(2)$  algebras in the product are equivalent to oscillator algebras whose Clebsch–Gordan coefficients are expressed in terms of Krawtchouk polynomials [21]. Note nevertheless, that even if we start with pure oscillator algebras (i.e.  $\mu_1 = \mu_2 = 0$ ), the addition rule is non-standard: it involves the reflection operator. Hence even in this simplest case the composed algebra will not be a pure oscillator algebra.

## 6 The Clebsch–Gordan problem in the Fock–Bargmann picture

The Clebsch–Gordan problem can be considered also in the Fock–Bargmann picture. This leads to a generating function for the Clebsch–Gordan coefficients.

The representation space for the coproduct is the space of polynomials in two variables  $f(x, y)$  which are homogeneous of degree  $N$ :

$$f(x, y) = y^N \Phi(x/y), \quad (6.1)$$

where  $\Phi(z)$  is a polynomial of degree  $N$  in the variable  $z$ .

For fixed  $N$  the action of the operator  $\tilde{J}_0$  is diagonal: it has the eigenvalue  $N + \mu_1 + \mu_2 + 1$  (due to Euler’s theorem on homogeneous polynomials).

Using the representation (6.1), we obtain the eigenvalue equation

$$\tilde{Q}f(x, y) = q_k f(x, y), \quad (6.2)$$

where the eigenvalues  $q_k$  are given by (5.3).

Substituting  $f(x, y)$  expressed as in (6.1) into (6.2) we obtain a differential-difference equation for the function  $\Phi(z)$ :

$$L\Phi_k(z) = q_k \Phi_k(z), \quad (6.3)$$

where the operator  $L$  is

$$\begin{aligned} L = & (-1)^N (z^2 + 1) \partial_z R + \left( (-1)^N \frac{\mu_1}{z} - (-1)^N (\mu_2 + N)z - \mu_1 - (-1)^N \mu_2 \right) R \\ & + \left( \mu_2 z - \frac{1}{2} - (-1)^N \frac{\mu_1}{z} \right) I, \end{aligned} \quad (6.4)$$

and where  $R$  acts according to  $R\Phi(z) = \Phi(-z)$  and  $I$  is the identity operator.

The operator  $L$  preserves the linear space of polynomials of degree  $\leq N$  and belongs to a class of Dunkl type operators of the first order considered in [22, 23, 24]. More precisely, the operator  $L$  is a linear combination (with coefficients depending on  $z$ ) of the operators  $I$ ,  $R$  and  $\partial_z R$ . The main difference with respect to the Dunkl type operators used in the papers mentioned above is that the operator (6.4) does not preserve the whole space of polynomials of a given arbitrary degree. Moreover, it is seen that the operator (6.4) is 3-diagonal in the monomial basis  $z^n$ ,  $n = 0, 1, \dots, N$ .

Using the decomposition of the function  $\Phi(z) = \Phi_e(z) + \Phi_o(z)$  into its even  $\Phi_e(z)$  and odd  $\Phi_o(z)$  parts we can reduce the equation (6.3) to standard hypergeometric equations for  $\Phi_e(z)$  and  $\Phi_o(z)$ .

The explicit form of the solution will depend on the parity of the integers  $N$  and  $k$ .

When both  $N$  and  $k$  even, we have

$$\begin{aligned}\Phi_k(z) = {}_2F_1 \left( \begin{matrix} -\frac{k}{2}, -\mu_2 - \frac{k}{2} + \frac{1}{2} \\ \mu_1 + \frac{1}{2} \end{matrix}; -z^2 \right) (1+z^2)^{\frac{N-k}{2}} \\ + \frac{kz}{2\mu_1+1} {}_2F_1 \left( \begin{matrix} 1 - \frac{k}{2}, -\mu_2 - \frac{k}{2} + \frac{1}{2} \\ \mu_1 + \frac{3}{2} \end{matrix}; -z^2 \right) (1+z^2)^{\frac{N-k}{2}},\end{aligned}$$

when  $N$  is even and  $k$  is odd

$$\begin{aligned}\Phi_k(z) = {}_2F_1 \left( \begin{matrix} -\frac{k+1}{2}, -\mu_2 - \frac{k}{2} \\ \mu_1 + \frac{1}{2} \end{matrix}; -z^2 \right) (1+z^2)^{\frac{N-k-1}{2}} \\ - \frac{2\mu_1 + 2\mu_2 + k + 1}{2\mu_1 + 1} z {}_2F_1 \left( \begin{matrix} -\frac{k-1}{2}, -\mu_2 - \frac{k}{2} \\ \mu_1 + \frac{3}{2} \end{matrix}; -z^2 \right) (1+z^2)^{\frac{N-k-1}{2}},\end{aligned}$$

for  $N$  odd and  $k$  even we have

$$\begin{aligned}\Phi_k(z) = {}_2F_1 \left( \begin{matrix} -\frac{k}{2}, -\mu_2 - \frac{k+1}{2} \\ \mu_1 + \frac{1}{2} \end{matrix}; -z^2 \right) (1+z^2)^{\frac{N-k-1}{2}} \\ + \frac{2\mu_1 + k + 1}{2\mu_1 + 1} z {}_2F_1 \left( \begin{matrix} -\frac{k}{2}, -\mu_2 - \frac{k-1}{2} \\ \mu_1 + \frac{3}{2} \end{matrix}; -z^2 \right) (1+z^2)^{\frac{N-k-1}{2}},\end{aligned}$$

finally for  $N$  and  $k$  odd

$$\begin{aligned}\Phi_k(z) = {}_2F_1 \left( \begin{matrix} -\frac{k-1}{2}, -\mu_2 - \frac{k}{2} \\ \mu_1 + \frac{1}{2} \end{matrix}; -z^2 \right) (1+z^2)^{\frac{N-k}{2}} \\ - \frac{2\mu_2 + k}{2\mu_1 + 1} z {}_2F_1 \left( \begin{matrix} -\frac{k-1}{2}, 1 - \mu_2 - \frac{k}{2} \\ \mu_1 + \frac{3}{2} \end{matrix}; -z^2 \right) (1+z^2)^{\frac{N-k}{2}}.\end{aligned}$$

(All these functions  $\Phi_k(z)$  are defined up to a common normalization factor.)

The solutions  $\Phi_k(z)$  are polynomials of degree  $N$  in  $z$ . It is clear from the definition (5.2) and (6.1) that the series expansion

$$\Phi_k(z) = \sum_{s=0}^N C_s^{(k)} z^s$$

gives the Clebsch–Gordan coefficients

$$C_s^{(k)} = W_{s;N,k}.$$

The polynomials  $\Phi_k(z)$  are thus generating functions for the Clebsch–Gordan coefficients of the  $sl_{-1}(2)$  algebra and hence for the dual  $-1$  Hahn polynomials.

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